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Perturbation Theory for Isotropic Landau-Lifschitz Equation Based on Inverse Scattering Transform[1]

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The perturbation theory for the Landau-Lifschitz equation for isotropic chain with correction, which is based on the inverse scattering transform (IST), is developed to treat Landau-Lifschitz equation for a spin chain with axis asymmetry. The time-evolution equation of parameters and a formula for the first-order correction is given by treating the equation with axis symmetry as a perturbation to the isotropic equation.

KEY WORDS: inverse scattering transform; perturbation theory.

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1. INTRODUCTION

Most of nonlinear equations can not be solved exactly directly, and a lot of methods are developed to give the solutions of the equations. Perturbation theory is one of them, which also extends the application of nonlinear equations. After inverse scattering transform was performed to solve the isotropic Landau-Lifschitz equation (Dodd *et al.*, 1982; Laksmanan, 1977; Fogedby, 1980), it remains a lot of works. The equation with axis asymmetry as well as full anisotropy haven't been solved completely by IST (Borovik, 1978; Bolovik and Kulinich, 1984). It seems solutions of these cases were not found exactly. Hence, we tried to treat the extra term as a perturbation to the isotropic equation. Two systematic perturbation methods, the method based on the inverse scattering transform (Kaup and Newell, 1978c; Karpman, 1979; Kivshar and Malomad, 1989) and the direct method based upon the theory of linear partial differential equations (Mjølhus, 1989; Mjølhus

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and Hada, 1997; Faddeev and Takhtajan, 1987; Chen *et al.*, 1998; Kivshar and Davies, 1998) have been well established and developed for completely and nearly integrable systems. The latter was firstly proposed by Mclaughlin and Scott in 1978 to deal with Sine-Gordon equation (Kaup and Newell, 1978a,b). At that time, in order to emphasize the difference between the solutions of direct perturbation theory and those based on the IST method, they tried to avoid the results derived from IST. Then the Green function method was produced to solve the perturbation (Gerdjikov *et al.*, 1980; Feng-Ming *et al.*, 2004; Hao and Nian-Ning, 2003). A series of problems in the perturbation theory of the nonlinear equations have been solved with the method mentioned above. We attempted to deal with perturbation of the L-L equation with the direct perturbation theory, however it is difficult to give the exact expression of the linearized operators because of the three parameters (S_1,S_2,S_3) of the spin chain. In this article, we construct the perturbation theory based on inverse scattering transform method. We treat the equation with axis symmetry as a perturbation to the isotropic equation. And we give the evolution equations of soltion parameters and the formula for calculating the first-order correction which can be studied further.

2. LANDAU-LIFSCHITZ EQUATION AND SOME RESULTS OF THE IST METHOD

The Landau-Lifschitz equation for a isotropic spin chain can be written as

$$\mathbf{S}_t + \mathbf{S} \times \mathbf{S}_{xx} = 0 \tag{1}$$

The Lax representations of the unperturbed isotropic equation are

$$\partial_x \psi(x, \lambda) = L\psi(x, \lambda), \quad L = -i\lambda S \cdot \sigma \tag{2}$$

And

$$\partial_t \psi(x, \lambda) = M\psi(x, \lambda), \quad M = -i2\lambda^2 S \cdot \sigma + i\lambda\sigma \cdot (S \times S_{xx}) \tag{3}$$

With asymptotic solutions of Eqs. (2) and (3), the usual Jost solutions are defined as

$$\Psi(x, \lambda) = (\tilde{\psi}(x, \lambda), \psi(x, \lambda)) \tag{4}$$

$$\Phi(x, \lambda) = (\phi(x, \lambda), \tilde{\phi}(x, \lambda)) \tag{5}$$

As usual, the monodramy matrix $T(\lambda)$ is introduced

$$\Phi(x, \lambda) = \Psi(x, \lambda)T(\lambda), \quad T(\lambda) = \begin{pmatrix} a(\lambda) & \tilde{b}(\lambda) \\ b(\lambda) & \tilde{a}(\lambda) \end{pmatrix} \tag{6}$$

$\psi(x, \lambda), \phi(x, \lambda)$ and $a(\lambda)$ are analytic in the upper half plane of complex λ; $\tilde{\psi}(x, \lambda)$, $\tilde{\phi}(x, \lambda)$ and $\tilde{a}(\lambda)$ are analytic in the lower half plane of complex λ. Usually $b(\lambda)$

and $\tilde{b}(\lambda)$ cannot be analytically continued outside the real axes. The Jost solutions in NLS$^+$ equation have some properties, such as

$$\tilde{\psi}(x,t,\bar{\lambda}) = -i\sigma_2\overline{\psi(x,t,\lambda)};\quad \tilde{\phi}(x,t,\bar{\lambda}) = i\sigma_2\overline{\phi(x,t,\lambda)} \tag{7}$$

and

$$\tilde{a}(\bar{\lambda}) = \overline{a(\lambda)},\quad \tilde{b}(\bar{\lambda}) = -\overline{b(\lambda)} \tag{8}$$

where $a(\lambda) = \prod_1^n \frac{\lambda-\lambda_n}{\lambda-\bar{\lambda}_n}$, when $a(\lambda)$ vanishing only at $\lambda = \lambda_n$, the usual IST method yields the single-soliton solution and the corresponding Jost solutions and their λ derivatives.

$$\psi_1(x,\lambda_1) = -\frac{1}{2}\frac{\lambda_1}{\bar{\lambda}_1}e^{i\lambda_1 x} sech\Theta e^{-i\Phi},\quad \psi_2(x,\lambda_1) = \frac{1}{2}e^{i\lambda_1 x} sech\Theta e^{\Theta} \tag{9}$$

$$\phi_1(x,\lambda_1) = \frac{1}{2}\frac{\lambda_1}{\bar{\lambda}_1}e^{-i\lambda_1 x} sech\Theta e^{-\Theta},\quad \phi_2(x,\lambda_1) = -\frac{1}{2}e^{-i\lambda_1 x} sech\Theta e^{i\Phi} \tag{10}$$

And their derivatives of λ

$$\begin{aligned}
\dot{\psi}_1(x,\lambda_1) &= i\left(x + \frac{1}{2\nu}\frac{\bar{\lambda}_1}{\lambda_1}\right)\psi_1(x,\lambda_1);\\
\dot{\psi}_2(x,\lambda_1) &= i\left(x + \frac{1}{2\nu}\frac{\bar{\lambda}_1}{\lambda_1}\right)\psi_2(x,\lambda_1) - i\frac{1}{2\nu}\frac{\bar{\lambda}_1}{\lambda_1}e^{i\lambda_1 x}
\end{aligned} \tag{11}$$

$$\begin{aligned}
\dot{\phi}_1(x,\lambda_1) &= -i\left(x - \frac{1}{2\nu}\frac{\bar{\lambda}_1}{\lambda_1}\right)\phi_1(x,\lambda_1) - i\frac{1}{2\nu}\frac{\bar{\lambda}_1}{\lambda_1}e^{-i\lambda_1 x},\\
\dot{\phi}_2(x,\lambda_1) &= -i\left(x - \frac{1}{2\nu}\frac{\bar{\lambda}_1}{\lambda_1}\right)\phi_2(x,\lambda_1)
\end{aligned} \tag{12}$$

In the equations above, we use the flowing expressions $\Phi = 2\mu x + 4(\mu^2 - \nu^2)t + \phi_{10}$, $\quad\Theta = 2\nu(x - x_1 + 4\mu t)$

In order to satisfy the second Lax Equation (3), the Jost solutions should be corrected as

$$\phi(x,t,\lambda) \to h(t,\lambda)\phi(x,t,\lambda),\quad \tilde{\phi}(x,t,\lambda) \to h^{-1}(t,\lambda)\tilde{\phi}(x,t,\lambda) \tag{13}$$

$$\tilde{\psi}(x,t,\lambda) \to h(t,\lambda)\tilde{\psi}(x,t,\lambda),\quad \psi(x,t,\lambda) \to h^{-1}(t,\lambda)\psi(x,t,\lambda) \tag{14}$$

Where $h(x,t,\lambda) = e^{-i2\lambda^2 t}$.

3. PERTURBATION THEORY BASED ON IST METHOD

The perturbed Landau-Lifschitz equation for isotropic spin chain can be written as

$$\mathbf{S}_t + \mathbf{S}\times\mathbf{S}_{xx} = \varepsilon\mathbf{P}(\mathbf{S}) \tag{15}$$

where ε is a small parameter and **P(S)** is a function of **S** characterizing the corrections.

The main idea of the perturbation theory base on the inverse scattering transform (IST) (Gerdjikov *et al.*, 1980) is to abandon the second Lax equation while preserving the first one. Only the last part of IST which decides the time dependence of the scattering date needs to be rebuilded. The new result should be distinguished to the origin one within a small quantity of the order of ε.

The L-L equation for a spin chain with easy axis can be written as

$$\mathbf{S}_t + \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times \mathbf{JS} = 0, \quad \mathbf{J} = \mathrm{diag}\,(0, 0, J), \quad J > 0 \tag{16}$$

If J is a small quantity, $J \to \varepsilon$, the last term of Eq. (16) can be treated as perturbation. Considering Eq. (15), have

$$\varepsilon \mathbf{P}(\mathbf{S}) = -\mathbf{S} \times \mathbf{JS} \tag{17}$$

Take $\varepsilon Q(s) = -i\lambda\sigma \cdot \varepsilon P(s)$, which is from the perturbation theory of IST (Kaup and Newell, 1978a). So

$$\varepsilon Q(s) = i\lambda\sigma \cdot (\mathbf{S} \times \mathbf{JS}) = -\lambda J \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \tag{18}$$

Considering the case of single soliton, $q = S_3(S_1 - iS_2) = \cos\theta \sin\theta e^{-i\varphi}$, $\cos\theta = 1 - \frac{2\nu^2}{(\mu^2+\nu^2)} sech^2\Theta$, $\varphi = \Phi + \arg\tanh(\frac{\nu}{\mu}\tanh\Theta)$.

From the perturbation theory of IST, the basic equations of the perturbation theory for isotropic L-L equation are obtained as follows

$$a_t(t, \lambda) = -\int_{-\infty}^{\infty} \psi(x, \lambda)^T i\sigma_2 G(x, \lambda)\phi(x, \lambda)\, dx \tag{19}$$

$$b_t(t, \lambda) - i4\lambda^2 b(\lambda) = \int_{-\infty}^{\infty} \tilde{\psi}(x, \lambda)^T i\sigma_2 G(x, \lambda)\phi(x, \lambda)\, dx \tag{20}$$

where $G(x, \lambda) = \{-i\lambda_t(S \cdot \sigma) + \epsilon Q(S)\}h(t, \lambda)\phi(x, \lambda)$, if $\varepsilon \to 0$, the result gives the one that the unperturbed equation gives.

For the bounded state, $a(\lambda_n) = 0$, $a_t(\lambda_n) = 0$ and $\phi(x, \lambda_n) = b_n(t, \lambda_n)$ $\psi(x, \lambda_n)$, These conditions indicates that the perturbation term is so small that it doesn't change the bounded state solution of the scattering problem as well as the soliton solution of the nonlinear equation. All the functions in (20) can be analytic continuously to the upper plane of the complex λ, therefore it's still valid in the limit of $\lambda \to \lambda_n$, so we have

$$\lambda_{nt} = \frac{i\varepsilon}{\dot{a}(\lambda_n)b_n(t)} \int_{-\infty}^{\infty} \phi(x, \lambda_n)^T \sigma_2 Q(x)\phi(x, \lambda_n)\, dx \tag{21}$$

Where we use the expression

$$\dot{a}(\lambda_n) = -i2 \int_{-\infty}^{\infty} \phi_1(x,\lambda)\phi_2(x,\lambda)dx \tag{22}$$

And

$$b_{nt}(t) - i4\lambda_n^2 b_n(t) = \frac{b_n(t)}{\dot{a}(\lambda_n)} \int_{-\infty}^{\infty} \{\dot{\phi}(x,\lambda_n) - b_n(t)\dot{\psi}(x,\lambda_n)\}^T \times i\sigma_2 G(x,\lambda_n)\psi(x,\lambda_n)dx \tag{23}$$

We can simplify the equation above by proofing the first part of the integration arising from the first part of $G(x,\lambda_n)$. In the case of discrete spectrum, $G(x,\lambda_n) = -i\lambda_{nt}\sigma \cdot S + \varepsilon Q(s)$, which is different from that of the continuous spectrum. So, Eq. (23) is reduced to

$$b_{nt}(t) - i4\lambda_n^2 b_n(t) = \frac{\varepsilon}{\dot{a}(\lambda_n)} \int_{-\infty}^{\infty} \{\dot{\phi}(x,\lambda_n) - b_n(t)\dot{\psi}(x,\lambda_n)\}^T \times i\sigma_2 Q(x)\phi(x,\lambda_n)dx \tag{24}$$

It is obviously that it gives the result as $\varepsilon \to 0$.

4. EFFECTS OF PERTURBATION ON THE SOLITON PARAMETERS

From Eq. (21), we can express $\lambda_1 = \mu + i\nu$, noticing $\dot{a}(\lambda_1) = \frac{1}{i2\nu}$ and $2\nu dx = d\Theta$, we have

$$\lambda_{1t} = i\lambda_1 J \int_{-\infty}^{\infty} \{q\psi_2(x,\lambda_1)\phi_2(x,\lambda_1) + \bar{q}\psi_1(x,\lambda_1)\phi_1(x,\lambda_1)\}d\Theta \tag{25}$$

Substituting the expressions of Jost solutions, taking $\eta = \arg\tan(\frac{\nu}{\mu}\tanh\Theta)$, $w = \arg\tan\frac{\nu}{\mu}$, it derives

$$\lambda_{1t} = -i\lambda_1 J \int_{-\infty}^{\infty} \frac{1}{8}\sin 2\theta sech^2\Theta\left(e^{\Theta}e^{-i\eta} + e^{-\Theta}e^{i(\eta+4\omega)}\right)d\Theta \tag{26}$$

In another hand, we derive the real and image part of the integration above.

$$\mu_t = J\frac{16}{15}\frac{\mu^2\nu(\mu^2-\nu^2)(5\mu^4 - 2\mu^2\nu^2 + \nu^4)|\nu|}{(\mu^2+\mu^2)^4|\mu|} \tag{27}$$

And

$$\nu_t = -J\frac{2}{15}\frac{\mu(\mu^2-\nu^2)(3\mu^2-\nu^2)(5\mu^4 - 10\mu^2\nu^2 + \nu^4)|\nu|}{(\mu^2+\mu^2)^4|\mu|} \tag{28}$$

Introduce two parameters ξ and δ which satisfy

$$\Theta = 2\nu(x-\xi), \quad \Phi = \frac{\mu}{\nu}\Theta + \delta \tag{29}$$

We know in the unperturbed case, $\xi = x_1 - 4\mu t$, $\delta = 2\mu x_1 - 4(\mu^2 + \nu^2)t + \phi_{10}$. With the new notations, $b_1(t, \lambda_1)$ can be written as

$$b_1(t, \lambda_1) = e^{-i(2\mu\xi - \delta + \pi)} e^{2\nu\xi} \tag{30}$$

Thus the left hand of Eq. (24) gives

$$-i2(\lambda_{1t} + \lambda_1 \xi_t + i\delta_t) - i4\lambda_1^2 \tag{31}$$

The integration of Eq. (24) gives ξ_t and δ_t by substituting the derivatives of the Jost solutions.

$$\xi_t = -4\mu + J\frac{4}{3}\frac{\mu^3\nu^2|\nu|}{(\mu^2+\nu^2)^3|\mu|} \tag{32}$$

And

$$\delta_t = -4(\mu^2 + \nu^2) + J\frac{2}{3}\frac{\mu^2(3\mu^2+\nu^2)|\nu|}{(\mu^2+\nu^2)^2|\mu|} \tag{33}$$

We can use the method of the derivative expansion, the independent variable t is transformed into several variables by

$$t_n = \varepsilon^n t \tag{34}$$

where each t_n is an order of ε smaller than the previous time. The time derivatives are replaced by the expansion

$$\partial_t = \sum_{n=0}^{\infty} \varepsilon^n \partial_{tn} \tag{35}$$

At the same time the dependent variable is expanded in an asymptotic series. As ε is a small quantity, we can express each parameter as the approximation of one order of ε

$$\xi = \xi_0 + \varepsilon t \xi_1, \quad \delta = \delta_0 + \varepsilon t \delta_1 \tag{36}$$

Consequently, we have

$$\mu = \mu_0 + J\frac{16}{15}\frac{\mu_0^2\nu_0\left(\mu_0^2 - \nu_0^2\right)\left(5\mu_0^4 - 2\mu_0^2\nu_0^2 + \nu_0^4\right)|\nu_0|}{\left(\mu_0^2 + \mu_0^2\right)^4|\mu_0|}t \tag{37}$$

$$\nu = \nu_0 - J\frac{2}{15}\frac{\mu_0\left(\mu_0^2 - \nu_0^2\right)\left(3\mu_0^2 - \nu_0^2\right)\left(5\mu_0^4 - 10\mu_0^2\nu_0^2 + \nu_0^4\right)|\nu_0|}{\left(\mu_0^2 + \mu_0^2\right)^4|\mu_0|}t \tag{38}$$

$$\xi = \xi_0 - 4\mu_0 t + J\frac{4}{3}\frac{\mu_0^3\nu_0^2|\nu_0|}{\left(\mu_0^2 + \nu_0^2\right)^3|\mu_0|}t \tag{39}$$

$$\delta = \delta_0 - 4\left(\mu_0^2 + \nu_0^2\right)t + J\frac{2}{3}\frac{\mu_0^2\left(3\mu_0^2 + \nu_0^2\right)|\nu_0|}{\left(\mu_0^2 + \nu_0^2\right)^2|\mu_0|}t \tag{40}$$

As $J \to 0$, the equations for the slow variation of the spectrum parameter reduce the unperturbed case.

5. THE FIRST-ORDER CORRECTION FOR THE ADIABATIC SOLUTION

By IST method we get the expression of the first-order correction of the adiabatic solution for isotropic L-L equation

$$\delta u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\lambda}_1}{\lambda'}r(\lambda')\psi_1^a(x,\lambda')^2 d\lambda' + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\lambda}_1}{\lambda'}\overline{r(\lambda')\psi_2^a}(x,\lambda')^2 d\lambda' \tag{41}$$

Where, $r(\lambda') = \frac{b(\lambda')}{a(\lambda')}$, $a(\lambda) = \frac{\lambda-\mu-i\nu}{\lambda-\mu+i\nu}$, $b(\lambda')$ is from the correction of the adiabatic solution which is valid in the case of reflectionless for unperturbed condition. And $b(\lambda')$ can be obtained from Eq. (20).

By some gauge transform (Huang, 1996) L-L equation can be equivalence to nonlinear Schrödinger (NLS) equation, therefore their Lax equations can be equivalence. The first Lax equation of NLS equation and L-L equation can be written respectively as

$$L_{NLS} = -i\lambda\sigma_3 + \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \tag{42}$$

And

$$L_{L-L} = -i\lambda(S\cdot\sigma) \tag{43}$$

Here, u is the adiabatic solution. Combining Eqs. (42) and (43), we have the result as following because of the equivalence between the adiabatic solution of NLS equation and that of L-L equation.

$$\begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} = i\lambda\begin{pmatrix} 1-S_3 & S_1 - iS_2 \\ S_1 + iS_2 & 1+S_3 \end{pmatrix} \tag{44}$$

So it is obtained the correction of the adiabatic solution, for the reason of $|S_1|^2 + |S_2|^2 + |S_3|^2 = 1$

$$\delta u = i\lambda(\delta S_1 - i\delta S_2) \tag{45}$$

From Eqs. (41) and (45) we can get δS_1 and δS_2.

6. CONCLUSION

In this paper, IST method is developed to study the perturbation theory of L-L equation with ease axis. The theory is applied to the description of soliton evolution in the presence of perturbation. It is shown a small perturbation leads to such effects as: (1) a slow change of soliton parameters; (2)a deformation of its shape. All these effects are investigated in detail for the Korteweg-de Vries, modified Korteweg-de Vries, and nonlinear Schrödinger equations to which perturbation terms of general form are added (Karpman and Maslov, 1978). In this article, We show the time dependence of the spectrum in the case of single soliton, by treating the asymmetry axis as a perturbation to isotropic spin chain, and we give the expression of the correction of the adiabatic solution of L-L equation. So we can consider the equation with easy plane by the same technique, which can also be extended to the applications of other nonlinear equations (Ao and Yan, 2005).

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